PMOR for Linear Systems with Many Inputs

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Modeling of physical systems often leads to parametrized linear time invariant (LTI) systems

(1) $$\mathbf{E}(\mu)\mathbf{x}' = \mathbf{A}(\mu)\mathbf{x} + \mathbf{B}(\mu)\mathbf{u}, \quad \mathbf{y} = \mathbf{C}(\mu)\mathbf{x} + \mathbf{D}(\mu)\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{x} = \mathbf{x}(\mu, t) \in \mathbb{R}^n$ is the state vector and $\mathbf{E}(\mu), \mathbf{A}(\mu) \in \mathbb{R}^{n\times n}, \mathbf{B}(\mu) \in \mathbb{R}^{n\times m}, \mathbf{C}(\mu) \in \mathbb{R}^{\ell\times n}, \mathbf{D}(\mu) \in \mathbb{R}^{\ell\times m}$. The vector $\mu \in \mathbb{R}^d$ represents the parameter variations which may arise from material properties, system configurations, etc. In practice, these systems have very large dimension compared to the number of inputs and outputs. Despite the ever increasing computational power, simulation of such systems in acceptable time is very difficult, in particular if multi-query tasks are required. This calls for application of parametric model order reduction (PMOR). PMOR replaces (1) by a parametric reduced-order model (pROM): $\mathbf{E}_r(\mu)\mathbf{x}_r' = \mathbf{A}_r(\mu)\mathbf{x}_r + \mathbf{B}_r(\mu)\mathbf{u}, \quad \mathbf{y}_r = \mathbf{C}_r(\mu)\mathbf{x}_r + \mathbf{D}(\mu)\mathbf{u}$, with the matrices $\mathbf{E}_r(\mu) = \mathbf{V}^{\mathrm{T}}\mathbf{E}(\mu)\mathbf{V}$, $\mathbf{A}_r(\mu) = \mathbf{V}^{\mathrm{T}}\mathbf{A}(\mu)\mathbf{V}$, $\mathbf{B}_r(\mu) = \mathbf{V}^{\mathrm{T}}\mathbf{B}(\mu), \mathbf{C}_r(\mu) = \mathbf{C}(\mu)\mathbf{V}$. The projection matrix $\mathbf{V} \in \mathbb{R}^{n\times r}$ which is valid for all parameters μ in the desired range, and for arbitrary inputs $\mathbf{u}$, can be constructed using, e.g., the implicit-moment matching PMOR method from [1]. However, direct application of this approach to system (1) with numerous inputs, i.e., m is very large, may produce pROMs with dense and large dimensional matrices, which are still computationally expensive. The same problem was already observed in non-parametric systems, and could be dealt with using the superposition principle, see [2].

We propose to apply the superposition principle to the parametrized system (1), which leads to m subsystems. Then, the standard PMOR method in [1] can be applied to each subsystem. Assume that $\mathbf{B}(\mu)$ has full column rank m, then can be split into $\mathbf{B}(\mu) = \sum_{i=1}^{m} \mathbf{B}_i(\mu)$, where $\mathbf{B}_i(\mu)$ are column rank-1 parametric matrices. By the superposition principle and using the above input matrix splitting, system (1) can be decomposed into m subsystems

(2) $$\mathbf{E}(\mu)\mathbf{x}_i' = \mathbf{A}(\mu)\mathbf{x}_i + \mathbf{B}_i(\mu)\mathbf{u}, \quad \mathbf{y}_i = \mathbf{C}(\mu)\mathbf{x}_i, \quad \mathbf{x}_1(0) = \mathbf{x}_0,$$

with $\mathbf{x}_i(0) = 0, i = 2, \ldots, m$. The output solution can be obtained through $\mathbf{y} = \sum_{i=1}^{m} \mathbf{y}_i + \mathbf{D}(\mu)\mathbf{u}$. If $\mathrm{blkdiag}(\mathbf{M})$ denotes the block-diagonal matrix with the matrix $\mathbf{M}$ on its diagonal, the parametrized system in (1) can be equivalently transformed into a block-diagonal system of dimension mn given by $\mathcal{E}(\mu)\tilde{\mathbf{x}}' = \mathcal{A}(\mu)\tilde{\mathbf{x}} + \mathcal{B}(\mu)\mathbf{u}, \quad \mathbf{y} = \mathcal{C}(\mu)\tilde{\mathbf{x}} + \mathbf{D}(\mu)\mathbf{u}$, where $\tilde{\mathbf{x}} = (\mathbf{x}_1^T, \ldots, \mathbf{x}_m^T)^T \in \mathbb{R}^{mn}$, $\mathcal{B}(\mu) = (\mathbf{B}_1(\mu)^T, \ldots, \mathbf{B}_m(\mu)^T)^T$, $\mathcal{C}(\mu) = (\mathbf{C}(\mu), \ldots, \mathbf{C}(\mu))$, $\mathcal{E}(\mu) = \mathrm{blkdiag}(\mathbf{E}(\mu), \ldots, \mathbf{E}(\mu))$ and $\mathcal{A}(\mu) = \mathrm{blkdiag}(\mathbf{A}(\mu), \ldots, \mathbf{A}(\mu))$. This system is an equivalent model of the original system in (1), in the sense that both produce the same output. Then, we approximate $\tilde{\mathbf{x}}$ by $\mathbf{V}\mathbf{x}_r$, with $\mathbf{V} = \mathrm{blkdiag}(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \ldots, \mathbf{V}^{(m)}) \in \mathbb{R}^{mn\times r}$, $r \ll n$, leading to the pROM of (1) given by: $\mathcal{E}_r(\mu)\mathbf{x}_r' = \mathcal{A}_r(\mu)\mathbf{x}_r + \mathcal{B}_r(\mu)\mathbf{u}, \quad \mathbf{y}_r = \mathcal{C}_r(\mu)\mathbf{x}_r + \mathbf{D}(\mu)\mathbf{u}$, with $\mathcal{E}_r(\mu) = \mathbf{V}^{\mathrm{T}}\mathcal{E}(\mu)\mathbf{V}$, $\mathcal{A}_r(\mu) = \mathbf{V}^{\mathrm{T}}\mathcal{A}(\mu)\mathbf{V}$, $\mathcal{B}_r(\mu) = \mathbf{V}^{\mathrm{T}}\mathcal{B}(\mu)$, $\mathcal{C}_r(\mu) = \mathcal{C}(\mu)\mathbf{V}$. The projection matrices $\mathbf{V}^{(i)}$ can be constructed by applying the PMOR method in [1] to each system in (2). The proposed PMOR method constructs much smaller and computationally cheaper pROMs than the standard PMOR methods.

Keywords: Parametric Model Order Reduction, Superposition Principle, Many Inputs.

References

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